For all questions, answer choice "E. NOTA" mean "None of the Above" answers is correct. Unless otherwise stated, assume NO figures are drawn to scale and length measurements are given in units.

1.

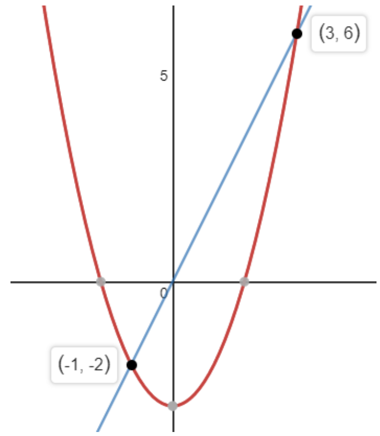
Solution:

A $f(x) = g(x)$ at $x = -1$ and 3. Thus the area of the shaded region between the two curves is

$$\int_{-1}^{3} f(x) - g(x)\,dx = \int_{-1}^{3} 2x - (x^2 - 3)\,dx$$

$$= -\frac{x^3}{3} + x^2 + 3x\Big|_{-1}^{3}$$

$$= 9 - \frac{1}{3} + 2 = \mathbf{\frac{32}{3}}$$



2.

Solution:

D The volume of sphere is $V = \frac{4}{3}\pi r^3$. Taking the derivative implicitly gives, $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$. Plugging in the given values, we have $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3(20)^2 \cdot 2 = \mathbf{3200\pi}$

3.

Solution:

D Let the side length of the cube be x and the radius of the sphere be r. Then $6x^2 = 4\pi r^2$, so $r = x\sqrt{\frac{3}{2\pi}}$. Substituting this value of r, the ration of the volume of a sphere to a cube is

$$\frac{\frac{4}{3}\pi r^3}{x^3} = \frac{\frac{4}{3}\pi\cdot\left(x\sqrt{\frac{3}{2\pi}}\right)^3}{x^3} = \frac{4\pi}{3}\cdot\frac{3}{2\pi}\sqrt{\frac{3}{2\pi}} = \sqrt{\mathbf{\frac{6}{\pi}}}$$

4.

Solution:

C Area $= \int_1^4 \left(\sqrt{x} - (-2x)\right) dx = \frac{2}{3}x^{3/2} + x^2\Big|_1^4 = \frac{2}{3}(4)^{3/2} + 16 - \frac{2}{3} - 1 = \mathbf{\frac{59}{3}}$.

5.

Solution:

B Since the distance from 1 to 6 is 5 units, each rectangle has a base width of 1 unit. Using a left-endpoint Riemann sum, the area approximation for $\int_1^6 x^2 + 2\,dx$ is equal to $A = bh = (1)[(1^2+2)+(2^2+2)+(3^2+2)+(4^2+2)+(5^2+2)] = \mathbf{65}$.

6.

Solution:

D The area can be calculated as $2 \cdot 2\pi - \int_0^\pi \frac{1}{2}\sin x\ dx = 4\pi - \left[-\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0\right] = \boldsymbol{4\pi - 1}$.

7.

Solution:

A The rate of volume increase as the height increases. Choices B and D can be eliminated since the volume will slow its increase as the height reaches the top of the container, where its diameter is decreasing. Choice C does not account for the slow in volume that will occur near the base of the container, where the diameter decreases.

8.

Solution:

B For a sphere, the rate of change of volume is $\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2$, and the rate of change of surface area is $\frac{dA}{dt} = 4\pi(2r)\frac{dr}{dt} = 8\pi r$. So, by setting $4\pi r^2 = 8\pi r$, we find that $r = \mathbf{2}$ will satisfy the equation.

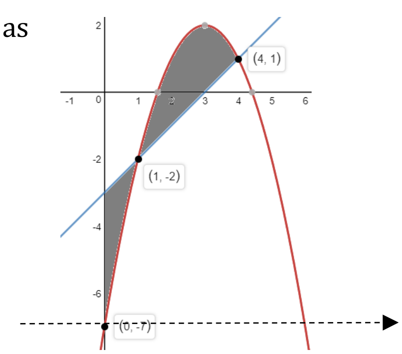
9.

Solution:

B To find the boundaries of the region, set $f(x) = g(x)$ and solve as

$$-(x-3)^2 + 2 = x - 3$$
$$-x^2 + 6x - x - 9 + 2 + 3 = 0$$
$$-x^2 + 5x - 4 = 0$$
$$(-x+1)(x-4) = 0$$

So, $x = 1$ or 4



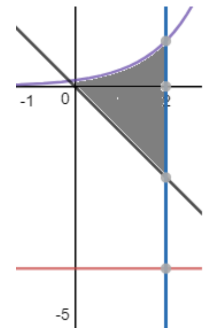
So, $V = \pi\int_0^1 (g(x) - (-7))^2 - (f(x) - (-7))^2\,dx +$
$\pi\int_1^4 (f(x) - (-7))^2 - (g(x) - (-7))^2\,dx$

$$\boldsymbol{V = \pi\int_0^1 (g(x)+7)^2 - (f(x)+7)^2\,dx + \pi\int_1^4 (f(x)+7)^2 - (g(x)+7)^2\,dx}$$

10.

Solution:

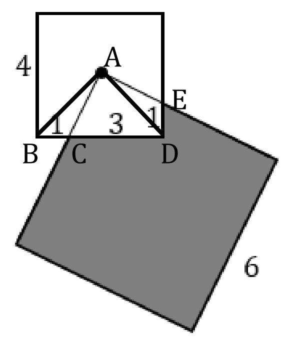
B The formula for the volume of a revolution of a solid produced by an enclosed region is $V = \pi \int_a^b (f(x) - k)^2 - (g(x) - k)^2 dx$, where $f(x)$ and $g(x)$ are the boundaries of the region revolved around the line $y = k$ from $x = a$ to $x = b$. Using this formula, we can plug in $f(x) = e^{x-2}$ and $g(x) = -x$, from $a = 0$ to $b = 2$ revolved about the line $y = -4$, giving $V = \pi \int_0^2 ((e^{x-2} + 4)^2 - (-x + 4)^2) dx$



11.

Solution:

D After drawing lines $\overline{AB}$ and $\overline{AD}$, it can be shown that $\Delta ABC \cong \Delta ADE$. Thus, the missing piece of the large square is ¼ the area of the small square. 36 – 4 = **32**.



12.

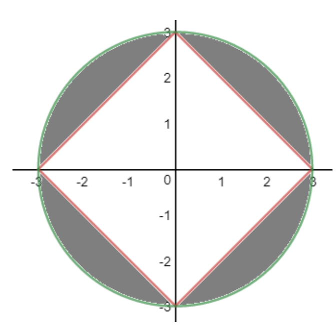
Solution:

B $9(x^2 - 4x + 4) + 36(y^2 + 8y + 16) = 36 + 576 - 288 = 324$. So, $a^2 = 36$ and $b^2 = 9$. Since the area of an ellipse is $\pi a \cdot b$, the area is $\pi 6 \cdot 3 = \mathbf{18\pi}$.

13.

Solution:

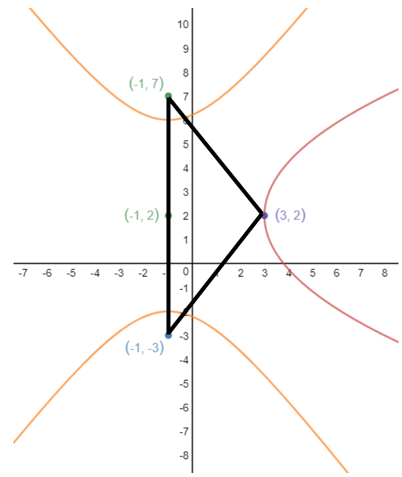
C As shown at right, the graph of $|x| + |y| = 3$ is a square with side lengths $3\sqrt{2}$, so the area of the square is 18. The graph of $x^2 + y^2 = 9$ is a circle with a radius of 3, so the area of the circle is 9π. Thus, the area between the graphs is $\mathbf{9\pi - 18}$.



14.

Solution:

B The parabola is $(y-2)^2 = 5(x-3)$, so the vertex is at (3, 2). The hyperbola is $\frac{(y-2)^2}{16} - \frac{(x+1)^2}{9} = 1$, so the center is (-1, 2). The distance from the center to the foci is $\sqrt{9+16} = 5$, which gives a distance of 10 between the two foci, or the base of the triangle. The height of the triangle is the distance between the vertex of the parabola and the center of the hyperbola, which is 4 units. Thus, the area of the triangle is $\frac{1}{2}(10)(4) = \mathbf{20}$.



15.

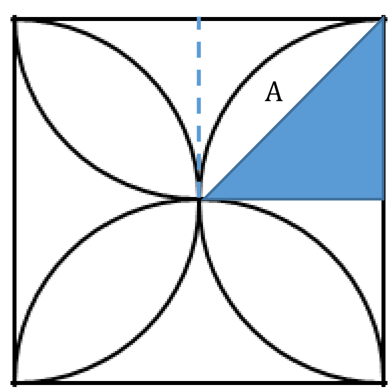
Solution:

C The volume is given by $V = \frac{1}{6}ansh$, where a is the apothem of the hexagon $(3\sqrt{3})$, n is the number of sides (6), s is the side length (6), and h is the height (15). So, $V = \frac{1}{6}(3\sqrt{3})(6)(6)(15) = \mathbf{270\sqrt{3}.}$

16.

Solutions:

A The flower is made up of 8 circular segments of radius 6 and angle 90°. One segment (marked A on figure at right) has an a quarter circle area of $\frac{36\pi}{4} = 9\pi$ minus half the area of the square section, $\frac{36}{2} = 18$. So, all 8 segments have an area, $8(9\pi - 18) = \mathbf{72(\pi - 2)}$.



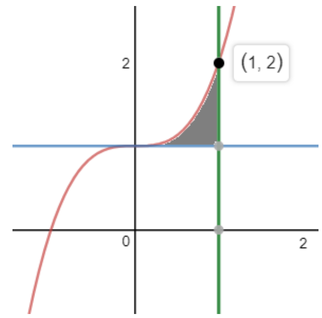
17.

Solution:

D Since the rotation is about the y-axis, to use the disk method the curve $y = x^3 + 1$ must be written as a function of y, as $x = \sqrt[3]{y-1}$.

$V = \pi \int_1^2 1^2 - \left(\sqrt[3]{y-1}\right)^2 dy = \pi \int_1^2 1 - (y-1)^{2/3}\, dy$

$= \pi \left[y - \frac{3}{5}(y-1)^{5/3}\right]_1^2 = \pi\left[2 - \frac{3}{5}\right] - \pi[1-0] = \mathbf{\frac{2\pi}{5}}$



18.

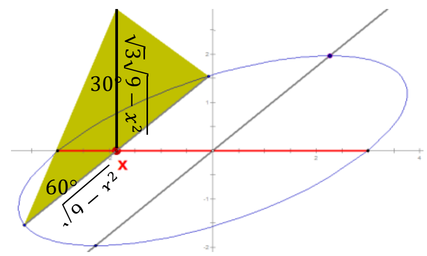
Solution:

A The length of the base of the triangle is $2y = 2\sqrt{9-x^2}$. The height of the triangle is $\sqrt{3}\sqrt{9-x^2}$. So, the area of a representative triangle is

$A = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{9-x^2})(\sqrt{3}\sqrt{9-x^2})$

$V = \int_{-3}^{3} \sqrt{3}\left(\sqrt{9-x^2}\right)^2 dx = \sqrt{3}\int_{-3}^{3}(9-x^2)dx =$

$\sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3} = \sqrt{3}\,[27 - 9 - (-27+9)] = \mathbf{36\sqrt{3}}$



19.

Solution:

B If the diagonal length of the box is 15, then if x, y, z are the side lengths of the box, it must be that $\sqrt{x^2+y^2+z^2} = 15$. So, $x^2+y^2+z^2 = 225$. By the inequality of arithmetic and geometric means, it follows that $\sqrt[3]{x^2y^2z^2} \le \frac{x^2+y^2+z^2}{3} = \frac{225}{3} = \mathbf{75^{3/2}}$.

20.

Solution:

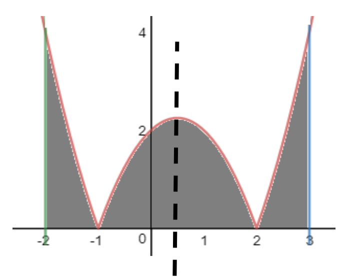
C The function $x^2 - x - 2$ has x-intercepts at $x = -1$ and 2. The area may then be found by integrating over 3 separate regions or using symmetry and doubling the integral value over 2 regions. The latter method is shown below:

$Area = 2\left[\int_{-2}^{-1} x^2 - x - 2\, dx + \int_{-1}^{1/2} -(x^2 - x - 2\,)dx\right]$

$= 2\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} + 2\left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{1/2}$

$= 2\left[\frac{-1}{3} - \frac{1}{2} + 2 - \left(\frac{-8}{3} - 2 + 4\right)\right] + 2\left[\frac{-1}{24} + \frac{1}{8} + 1 - \left(\frac{1}{3} + \frac{1}{2} - 2\right)\right]$

$= 2\left[-\frac{5}{6} + \frac{8}{3}\right] + 2\left[\frac{1}{12} - \frac{5}{6} + 3\right] = \frac{22}{6} + \frac{27}{6} = \mathbf{\frac{49}{6}}$.



21.

Solution:

D The two curves intersect where $2x+1=x^2-2$, which is at $x=-1$ and 3. So, the area can be calculated as $Area=\int_{-1}^{3} 2x+1-(x^2-2)\,dx=\int_{-1}^{3} -x^2+2x+3\,dx$

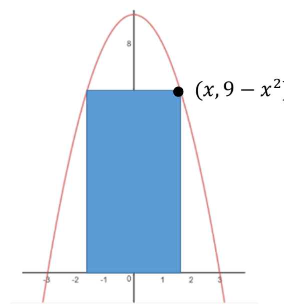
$$= -\frac{x^3}{3}+x^2+3x\Big|_{-1}^{3} = -\frac{3^3}{3}+3^2+3(3)-\left[\frac{1}{3}+1-3\right]=\frac{32}{3}=\frac{2^5}{3}$$

From this, we find that $a=5$ and $b=1$, so $a(b+a)=5(1+5)=$ **30**.

22.

Solution:

A From the figure at right, the point $(x,9-x^2)$ represents half the rectangle width, x, and the length, $9-x^2$. The area of the rectangle can be written as $A=lw=2x(9-x^2)=18x-2x^3$. To find the maximum dimensions, take the derivative and find the critical points, $A'=18-6x^2=0$, $18=6x^2$, so $x=\pm\sqrt{3}$. Realistically, the dimension for x has to be positive, and the largest rectangular area is $2\sqrt{3}\times 6$.



23.

Solution:

C In polar, the line $y=x$ is equal to $\theta=\frac{\pi}{4}$, and the line $y=-x$ is equal to $\theta=-\frac{\pi}{4}$ or $\frac{3\pi}{4}$. Because the graph of $r=3\sin\theta$ is a circle in Quadrants I and II, the enclosed area will be contained between the angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, giving $A=\frac{1}{2}\int_{\pi/4}^{3\pi/4}(3\sin\theta)^2\,d\theta=\boldsymbol{\frac{9}{2}\int_{\pi/4}^{3\pi/4}\sin^2\theta\,d\theta}$.

24.

Solution:

E To set up this related rate problem, we first need to find a proportional relationship between the radius and height of the cone, which is $\frac{r}{h}=\frac{2}{6}$ so $r=\frac{h}{3}$. Using the volume of a cone formula, we substitute this value of r as $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h=\frac{1}{27}\pi h^3$ and take the derivative implicitly on both sides as,

$$\frac{dV}{dt}=\frac{1}{9}\pi h^2\frac{dh}{dt}\rightarrow\frac{1}{9}\pi(4)^2\frac{dh}{dt}=\left(\frac{3}{4}\right)\rightarrow\boldsymbol{\frac{27}{64\pi}\ in/sec}.$$

25.

Solution:

A Taking the derivative of $x^3 - 2xy^2 + y^3 - 1 = 0$ gives

$$3x^2\frac{dx}{dt} - \left[2x \cdot 2y\frac{dy}{dt} + y^2 \cdot 2\frac{dx}{dt}\right] + 3y^2\frac{dy}{dt} = 0$$

Substituting $x = 1, y = 2$ and $\frac{dx}{dt} = 3$, we have

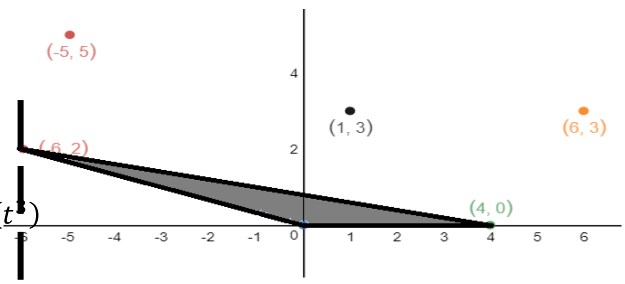
$$9 - \left[8\frac{dy}{dt} + 24\right] + 12\frac{dy}{dt} = 0 \rightarrow 4\frac{dy}{dt} = 15 \rightarrow \frac{dy}{dt} = \frac{15}{4}$$ units per second.

So, the area of the rectangle is $A = xy \rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = 1\left(\frac{15}{4}\right) + 2(3) = \mathbf{\frac{39}{4}}$.

26.

Solution:

E For convenience, translate the triangle so point B is at the origin, giving new points as A=(-6,2), B=(0,0), and C=(4, 0). The equation for the line AB is $y = -\frac{x}{3}$ or $-3y = x$, and the equation for line AC is $y = -\frac{x}{5} + \frac{4}{5}$ or $-5y + 4 = x$. Using the $\cos(t^3)$ disc method and revolving around the line $x = -6$ gives a volume of $V = \pi\int_0^2 (4 - 5y + 6)^2 - (-3y + 6)^2 dy$



$$V = \pi\int_0^2 (10 - 5y)^2 - (6 - 3y)^2 dy = \pi\int_0^2 64 - 64y + 16y^2\, dy$$

$$= \pi\left[64y - 32y^2 + \frac{16}{3}y^3\right]_0^2 = \mathbf{\frac{128}{3}\pi}.$$

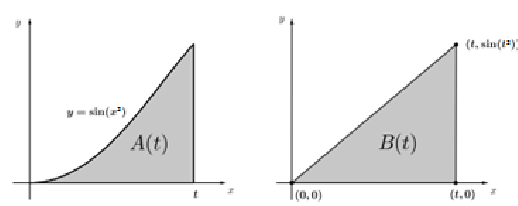
27.

Solution:

C $A(t) = \int_0^t \sin(x^3)dx, \quad B(t) = \frac{1}{2}t\sin(t^3)$

$\lim_{t\to0^+} A(t) = 0$ *and* $\lim_{t\to0^+} B(t) = 0$, so by L'Hospital's Rule $\lim_{t\to0^+}\frac{B(t)}{A(t)} = \lim_{t\to0^+}\frac{B\prime(t)}{A\prime(t)}$.

Using L'Hospital's Rule twice gives



$$\lim_{t\to0^+}\frac{\frac{3}{2}t^3\cos(t^3) + \frac{1}{2}\sin(t^3)}{\sin(t^3)} = \lim_{t\to0^+}\frac{-3t^3\sin(t^3) + 4\cos(t^3)}{2\cos(t^3)} = 2$$

28.

Solution:

C $\int_a^\infty \frac{8}{x^2}dx = \frac{1}{2}\int_1^\infty \frac{8}{x^2}dx = 4$, so $a = 2$

29.

Solution:

D The curves intersect when $2 + \cos 2\theta = 2$, which gives $\cos 2\theta = 0$. So, $\theta = \frac{\pi}{4}$.

The area of the shaded region in Quadrant 1 can be found by subtracting the area accumulated on $r = 2 + \cos 2\theta$ from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ from the circular area over the same interval.

$$A = 4\left[\frac{1}{2}\int_{\pi/4}^{\pi/2}(2)^2 d\theta - \frac{1}{2}\int_{\pi/4}^{\pi/2}(2 + \cos 2\theta)^2 d\theta\right] = 2\pi - 2\int_{\pi/4}^{\pi/2}(2 + \cos 2\theta)^2 d\theta$$

$$= 2\pi - 2\int_{\pi/4}^{\pi/2} 4 + 4\cos 2\theta + \cos^2 2\theta \, d\theta = 2\left[4\theta + 2\sin 2\theta + \left(\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right)^*\right]_{\pi/4}^{\pi/2}$$

$$= 2\pi - 2\left[2\pi + \frac{\pi}{4} - \pi - 2 - \frac{\pi}{8}\right] = \mathbf{4 - \frac{\pi}{4}}.$$

*Use of power reducing formula

30.

Solution:

B The volume of the solid is $V = \pi\int_1^\infty x^2 e^{-2x}\, dx = \pi \lim_{b\to\infty}\int_1^b x^2 e^{-2x}\, dx$

Integrating by parts two times gives the following:

$$\int x^2 e^{-2x}\, dx = -\frac{x^2}{2}e^{-2x} + \int x e^{-2x}\, dx$$

$$= -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\, dx$$

$$= -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$= -\frac{2x^2+2x+1}{4e^{2x}} + C$$

So, $V = \pi \lim_{b\to\infty}\left[-\frac{2x^2+2x+1}{4e^{2x}}\right]_1^b = \pi \lim_{b\to\infty}\left[-\frac{2b^2+2b+1}{4e^{2b}} + \frac{5}{4e^2}\right]_1^b = \frac{5\pi}{4e^2}$

and the volume of the solid is $\mathbf{\frac{5\pi}{4e^2}}$.